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BRANCHINGS IN ROOTED GRAPHS AND THE DIAMETER OF GREEDOIDS

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Let G be a 2-connected rooted graph of rank r and A, B two (rooted) spanning trees of G We show that the maximum number of exchanges of leaves that can be required to transform A into B is r^2-r+1 $(r>0)$. This answers a question by L. Lovász.

There is a natural reformulation of this problem in the theory of *greedoids*, which asks for the maximum diameter of the basis graph of a 2-connected branching greedoid of rank r.

Greedoids are finite accessible set systems satisfying the matroid exchange axiom. Their theory provides both language and conceptual framework for the proof. However, it is shown that for general 2-connected greedoids (not necessarily constructed from branchings in rooted graphs) the maximum diameter is 2^r-1.

1. Introduction

The following question was posed by László Lovász: how many steps are at most required for changing one spanning tree of a 2-connected rooted graph into a given second one, if every step consists of the exchange of a single leaf? We show that the answer is r^2-r+1, where r denotes the rank of the graph.

Both question and answer are conveniently stated in terms of greedoids, where the problem asks for the maximal diameter of the basis graph of a 2-connected branching greedoid. We get the answer (Theorem 4.6) as part of a general study of the diameter of basis graphs and basic word graphs for different classes of greedoids.

Greedoids are finite accessible set systems satisfying the matroid exchange axiom. They are a generalization of matroids introduced by Korte and Lovász [7] as the common structure underlying many greedy algorithms. In fact greedoids can be characterized as set systems by the optimality of the greedy algorithm for a class of objective functions, for which breadth-first search in graphs and certain scheduling problems under precedence constraints are examples. We refer to [7] and [9] for details.

Here we consider a different type of problems and algorithms on greedoids, which concern "pivoting steps" (one element exchanges with feasibility constraint) between different solutions (bases). The complexity of these algorithms is measured by the diameter of the basis graph [10]. In this graph two bases are connected by an edge if they differ by only one pivoting step.

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The paper is organized as follows. In Section 2 we give definitions and basic properties of greedoids to be fairly self-contained. (See [13] for a broader introduction to greedoids.) We discuss the concept of connectivity of greedoids and its compatibility with basic greedoid constructions.

In Section 3 we show that for 2-connected greedoids the diameter of the basis graph can grow exponentially with the rank. However, in the case of greedoids corresponding to search in graphs ("branching greedoids") the diameter of the basis graph is bounded by a quadratic function in the rank (Section 4). In both cases we can determine the exact bounds and construct the corresponding extremal examples.

In Section 5 we discuss the case of higher connectivity. For this it is necessary to assume the interval property, which is, however, shared by all major classes of examples. Greedoids of higher connectivity have been studied in a topological setting by Björner, Korte and Lovász [4].

In turns out that higher connectivity of the greedoids decreases the possible diameter. For maximally connected greedoids we prove a linear bound for the diameter of the basis graph. We conjecture that for greedoids of connectivity at least three a quadratic bound applies.

All the cases mentioned will also be discussed for the basic word graphs, which arise naturally from the definition of greedoids as left hereditary simple exchange languages. The diameter of the basic word graphs is of interest because it appears to be the natural measure of complexity for algorithms to find paths between given bases of a greedoid.

Comparing the results for different classes of greedoids, we see that the diameter is a quite sensitive parameter for the complexity of greedoids and the exchange algorithms modeled on them.

2. Greedoids and connectivity

Definition 2.1. A *greedoid* $\mathscr{F}$ on a finite set E is a pair $(E, \mathscr{F})$ where $\mathscr{F}$ is a nonempty collection of subsets of E satisfying:

(G1) For X in $\mathscr{F}$ nonempty, there is an $x \in X$ such that $X \setminus \{x\}$ is in $\mathscr{F}$.

(G2) For X, Y in $\mathscr{F}$, $|X| > |Y|$, there is an $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in $\mathscr{F}$.

(G1) states that $\mathscr{F}$ is an *accessible set system*, (G2) is the exchange axiom for the independent sets of a matroid. The sets in $\mathscr{F}$ are called *feasible*, their cardinality will be called *rank*. Thus $\mathscr{F}$ is a matroid if and only if every subset of a feasible (independent) set is again feasible. The maximal feasible sets of a greedoid are called *bases*.

As usual the matroid exchange axiom (G2) implies that all the bases of a greedoid have the same cardinality, which we call the *rank* r of the greedoid. The rank will be our principal parameter for the size of a greedoid.

For an arbitrary subset A of the ground set E we define its *rank* by $r(A) = \max \{|X|: X \subseteq A, X \in \mathscr{F}\}$, its *corank* by $cr(A) = r - r(A)$. Thus $A \subseteq E$ is feasible iff $r(A) = |A|$, it is a basis iff $r(A) = |A| = r$. Feasible sets of corank 1 (rank $r-1$) will be called *subbases*.

A greedoid is *full* if $E \in \mathscr{F}$. For convenience we often do not mention E explicitly, assuming $E = \bigcup \mathscr{F}$ for a given finite set system $\mathscr{F}$. Then we call $\mathscr{F}$ full if it has only one basis.

If we order $\mathcal{F}$ by inclusion, we get a finite, graded poset (partially ordered set) with minimal element $\hat{0}=\emptyset$. We use the Hasse diagram of $\mathcal{F}$ to get a graphic representation of $\mathcal{F}$. However the Hasse diagram has to be labeled, as the greedoid structure of $\mathcal{F}$ can not be reconstructed from the partial order (compare Figure 1). Björner [2] discusses the poset representations of greedoids, of which the "canonical" representation by $\mathcal{F}$, ordered by inclusion, and the universal representation by the poset of flats (which we will use extensively in Section 4) are examples. We refer to [2] for details.

There is an equivalent ordered version of greedoids in terms of exchange languages, which we describe now. For a finite set E, let E^* be the free monoid of *words* in the alphabet E. We use greek letters α, β, γ for words in E^*, latin letters x, y, z for letters, i.e. elements of E. $|\alpha|$ will denote the length of α, i.e. the number of (not necessarily distinct) letters in α. The *support* $\tilde{\alpha}$ of α is the set of letters of α. A word α is called *simple* if it does not contain any letter twice, that is $|\alpha|=|\tilde{\alpha}|$. Now a *language* $\mathcal{L}$ over E is a nonempty subset $\mathcal{L}\subseteq E^*$; it is called *simple* if every word in $\mathcal{L}$ is simple. By the *support* $\tilde{\mathcal{L}}$ of the language $\mathcal{L}$ we mean $\tilde{\mathcal{L}}=\{\tilde{\alpha}: \alpha\in\mathcal{L}\}$.

Definition 2.2. A *greedoid language* $\mathcal{L}$ on a finite ground set E is a finite simple language $\mathcal{L}\subseteq E^*$ satisfying:

(G1′) If $\alpha=\beta\gamma$, $\alpha\in\mathcal{L}$, then $\beta\in\mathcal{L}$, i.e. every beginning section of a word in $\mathcal{L}$ is again a word in $\mathcal{L}$.

(G2′) If $\alpha, \beta\in\mathcal{L}$, $|\alpha|>|\beta|$ then there is an $x\in\alpha$ such that $\beta x\in\mathcal{L}$.

(G1′) states that $\mathcal{L}$ is a *left hereditary language*, (G2′) is an exchange axiom. The maximal words in $\mathcal{L}$ are called *basic words*, their (common) length is the *rank* of $\mathcal{L}$.

It is easy to see that the Definitions 2.1 and 2.2 are actually equivalent: if $\mathcal{L}$ is a greedoid language as defined in 2.2, then $\tilde{\mathcal{L}}$ satisfies (G1) and (G2). Conversely, given a greedoid $\mathcal{F}$, we can (re-)construct $\mathcal{L}$ as

$$\mathcal{L}=\{x_1x_2\dots x_n\colon \{x_1, x_2, \dots, x_i\}\in\mathcal{F},\quad \left|\{x_1, x_2, \dots, x_i\}\right|=i \quad\text{for}\quad 1\leqq i\leqq n\}.$$

Thus it is not misleading if we occasionally use the term "greedoid" for greedoid languages, too.

The following constructions with greedoids will be considered. Given a greedoid $\mathcal{F}$ of rank r, then the *k-truncation* of $\mathcal{F}$ is the greedoid $\mathcal{F}|_k=\{X\in\mathcal{F}\colon r(X)\leqq k\}$. This corresponds to truncating the poset $\mathcal{F}$ at rank k. For a feasible set $A\in\mathcal{F}$ we define the *contraction* of $\mathcal{F}$ by A as the greedoid $\mathcal{F}/A=\{X\setminus A\colon X\in\mathcal{F}$ and $A\subseteq X\}$. This corresponds to taking the principal filter generated by A in the poset $\mathcal{F}$. In both cases it is easy to verify that the defined structures are actually greedoids. Note, however, that $\mathcal{F}/A$ is not a greedoid if A is not feasible.

By a t-minor of $\mathcal{F}$ we will mean the truncation of a contraction or, equivalently, the contraction of a truncation of $\mathcal{F}$. This differs from the general usage where minors denote contractions of restrictions. The *restriction* of $\mathcal{F}$ to $A\subseteq E$ is the greedoid $\mathcal{F}|A=\{X\in\mathcal{F}\colon X\subseteq A\}$. However we will observe in 2.5 that the connectivity of greedoids is badly behaved with respect to restrictions. Thus — as diameter questions under connectivity constraints are the principal object of study for this paper — t-minors are more suited to construct substructures in a given greedoid than minors.

In poset language t-minors are truncations of principal filters. If we use only restrictions to feasible sets, then the corresponding minors are intervals in $\mathcal{F}$. The general case is more complicated.

A *subgreedoid* of a greedoid $(E, \mathcal{F})$ is a greedoid $(E_0, \mathcal{F}_0)$ such that $E_0 \subseteq E$, $F_0 \subseteq F$ and $\text{rank}(F_0) = \text{rank}(F)$. This is not the most general definition possible, but will be suitable for our purposes (uniqueness results in Proposition 4.7). For example matroid slimmings in the sense of [9] are subgreedoids, but truncations and contractions of a greedoid are not subgreedoids (except for the trivial cases).

We are now going to define connectivity in greedoids and related concepts. Our definitions are slightly more general then the usual ones from [4], generalizing to arbitrary graded posets with $\hat{0}$. They specialize to the definitions in [4] if applied to the poset $\mathcal{F}$ of a greedoid.

Let $\mathcal{P}$ be a finite graded poset of rank r with minimal element $\hat{0}$. For $X \in \mathcal{P}$ we call a set A of covers *X-free* if A is the set of atoms of a boolean graded subposet of $\mathcal{P}$ with minimal element X, that is there is a $Y \in \mathcal{P}$, $Y \geqq X$, $r(X, Y) = |A|$, such that $[X, Y]$ contains a boolean algebra with A as its set of atoms. Particularly two covers a_1, a_2 of X are *independent over* X if $\{a_1, a_2\}$ is free over X, that is if there is a $Y \in \mathcal{P}$ which covers a_1 and a_2.

Definition 2.3. A finite graded poset $\mathcal{P}$ with $\hat{0}$ is *k-connected* if for every $X \in \mathcal{P}$ there is an X-free set of covers of size at least $\min\{k; r - r(X)\}$.

Note that every $\mathcal{P}$ is 1-connected, and "$(k+1)$-connected" implies "k-connected".

The connectivity of a poset can be easily read off from its Hasse diagram. This also yields a way of checking the connectivity of a given greedoid, ordered by inclusion. It pays off, however, to translate the poset definitions into concepts in terms of set systems.

Given a greedoid $(E, \mathcal{F})$, a subset A of E will be called *free over* $X \in \mathcal{F}$ if $A \cap X = \emptyset$ and for every $B \subseteq A$: $X \cup B \in \mathcal{F}$. If A is free over X, then obviously every subset of A is again free over X. Especially $\{x\}$ is free over X if $x \notin X$ and $X \cup \{x\} \in \mathcal{F}$. Then we call x a *continuation* of X, which corresponds to a cover of X in the poset $\mathcal{F}$. $\{x, y\}$ free over X means that $X \cup \{x\}$, $X \cup \{y\}$ and $X \cup \{x, y\}$ are feasible and distinct. We then call x and y *independent over* X. Now $\mathcal{F}$ will be called *2-connected* if for every $X \in \mathcal{F}$ of corank at least 2, there is a free 2-set over X, i.e. X has at least two independent continuations. In general:

Definition 2.3'. A greedoid $\mathcal{F}$ is *k-connected* if for every $X \in \mathcal{F}$, there is a free set of size $\min\{k; \text{cr}(X)\}$ over X, i.e. there is a set $A \subseteq E \setminus X$, $|A| = \min\{k; \text{cr}(X)\}$ such that $X \cup B \in \mathcal{F}$ for every $B \subseteq A$.

Again it is clear that every greedoid is 1-connected, and for $k=2$ our definitions coincide. Observe that 2.3' specializes 2.3: a greedoid $\mathcal{F}$ is k-connected if and only if the poset $\mathcal{F}$ is k-connected.

Additionally any truncation or contraction of a k-connected greedoid is again k-connected. Thus k-connectivity is preserved under taking t-minors. However, for $k > 2$ the study of k-connected greedoids usually requires an extra condition, the interval property:

Definition 2.4. A greedoid has the *interval property*, if for $X, Y, Z \in \mathcal{F}$, $X \subseteq Y \subseteq Z$, $x \notin Z$, always $X \cup \{x\} \in \mathcal{F}$ and $Z \cup \{x\} \in \mathcal{F}$ imply that $Y \cup \{x\} \in \mathcal{F}$.

A greedoid with the interval property will be called an *interval greedoid.* Most of the interesting classes of greedoids (and all the classes discussed in this paper) have this property. See [7] for details. A special case is given by the *upper interval greedoids,* for which $X, Y \in \mathcal{F}, X \subseteq Y, a \notin Y$ and $X \cup \{a\} \in \mathcal{F}$ imply $Y \cup \{a\} \in \mathcal{F}$. Equivalently, these are the greedoids such that the union of feasible sets is always feasible. They are also known as *APS-greedoids, convex geometries, shelling structures* or *antimatroids.* See [2] and [6] for further discussion.

We now observe that k-connectivity is not preserved under restrictions, even if we consider only $k=2$ and only restrictions to feasible sets. In fact we have:

Proposition 2.5. Let $\mathcal{F}$ be an interval greedoid such that the restriction to every feasible set is 2-connected. Then $\mathcal{F}$ is a matroid.

Proof. Let B be a basis of $\mathcal{F}$, then we have to show that the interval $I=[\emptyset, B]$ in $\mathcal{F}$ is boolean. I is a subposet of the boolean algebra 2^B. It is a semimodular lattice by [4, Thm. 3.2.]. Now 2-connectivity of the restrictions implies that every interval of rank 2 in I has four elements, hence by [3] I is relatively complemented. Thus I is a geometric lattice, especially atomic. Hence all the elements in B are feasible, and we are done by the interval property [4, Lemma 2.2.]. ▮

Finally we define the graphs whose diameters we use as a measure for the complexity of basis exchange algorithms. Two bases A and B of a greedoid $\mathcal{F}$ are called *adjacent* if $|A \cap B| = |A|-1$ and $A \cap B \in \mathcal{F}$, that is A and B differ in exactly one element and their intersection is feasible. This gives rise to the *basis graph* $G(\mathcal{F})$ (or simply G) on the set of bases of $\mathcal{F}$, whose edges are pairs of adjacent bases.

A very similar construction yields the *subbasis graph* $\bar{G}$ which will turn out to be useful in different contexts: in this graph two subbases X and Y of $\mathcal{F}$ are joined by an edge if their union is a basis, that is $|X \cup Y| = |X|+1$ and $X \cup Y \in \mathcal{F}$.

Two basic words α and β of a greedoid language $\mathcal{L}$ are *adjacent* if β is obtained from α by interchanging two consecutive letters or by changing the last letter. Equivalently, α and β are adjacent if the corresponding maximal chains in the poset $\mathcal{F}=\tilde{\mathcal{L}}$ differ by exactly one element. This notion of adjacency defines the *basic word graph* $\hat{G}$ on the set of basic words of $\mathcal{L}$. There is an obvious graph map from the basic word graph $\hat{G}$ of $\mathcal{L}$ to the basis graph G of the corresponding greedoid $\tilde{\mathcal{L}}$, sending every basic word α to its support $\tilde{\alpha}$. Hence for every greedoid $\operatorname{diam} G(\mathcal{F}) \leq \operatorname{diam} \hat{G}(\mathcal{F})$.

We stress the fact that basis graph, subbasis graph and basic word graph only depend on the poset $\mathcal{F}$, not on the actual greedoid structure. In fact the basis graph and the subbasis graph of a greedoid arise as the "intersection graphs" of the bipartite graph given by the two top rank levels of the poset $\mathcal{F}$ (having bases and subbases as vertices and cover relations as edges). This implies

Lemma 2.6. $|\operatorname{diam}(G) - \operatorname{diam}(\bar{G})| \leq 1$. Especially $\bar{G}$ is connected if and only if G is connected. ▮

In [10], Korte and Lovász prove that 2-connectivity of $\mathcal{F}$ implies the basis graph of $\mathcal{F}$ to be connected. In fact they show that in this case $\hat{G}$ is connected, and the graph map $\hat{G}\to G$ implies that G is connected. This result will also be a corollary of our Theorem 3.1.

Note that the following (weaker) converse is true: call $\mathscr{F}$ *nearly 2-connected* if for every $X\in\mathscr{F}$, $\operatorname{cr}(X)\geqq 2$, with at least two continuations, there is a free 2-set over X. Obviously every 2-connected greedoid is nearly 2-connected, and we have

Proposition 2.7. All the t-minors of a greedoid $\mathscr{F}$ have connected basis graph if and only if $\mathscr{F}$ is nearly 2-connected.

Proof. It is sufficient to prove that the basis graph of every nearly 2-connected greedoid is connected, because nearly 2-connectedness is preserved under taking t-minors. But for this Corollary 3.2 of Theorem 3.1 applies. Note that Theorem 3.1 and its lemmas are formulated only for 2-connected greedoids, however everything stays valid if we weaken the hypotheses to nearly 2-connectedness. ∎

In the following we shall work with 2-connected rather than with nearly 2-connected greedoids, mostly for convenience. In all nondegenerate situations, that is if every feasible set of corank at least 2 has at least two covers (continuations), the two concepts are equivalent anyway. However we note the following:

Lemma 2.8. Upper interval greedoids are nearly 2-connected. Their basis graphs are trivial, subbasis graphs are complete. Their basic word graphs and the basis graphs of their truncations are connected.

Proof. $\mathscr{F}$ is a semimodular lattice by [4]. Hence the first two statements are trivial. The rest can be seen from the shellability of semimodular lattices [1] or even more easily from Proposition 2.7; it is also proved in [10]. ∎

We refer to Korte and Lovász [7], [8] and [3] for further discussion of structural properties and examples of greedoids, mentioning here only a few further classes of special interest for the following discussions:

Let D be a finite, rooted, directed graph with edge set E, and $\mathscr{F}$ the set of trees in D that contain the root and are directed away from it. Then $\mathscr{F}$ is a *(directed) branching greedoid* or *search greedoid* [7]. If we assume without loss of generality that D admits a spanning tree, then $\mathscr{F}$ is k-connected exactly if D is k-connected as a digraph, that is no vertex can be separated from the root by less than k vertices, or (equivalently) there are k disjoint directed paths from the root to any vertex not adjacent to it.

Undirected branching greedoids are defined analogously on undirected, rooted graphs. Note that for an undirected branching greedoid $(E,\mathscr{F})$ the standard construction of replacing edges in the graph by pairs of antiparallel arcs yields a canonical directed branching greedoid $(E',\mathscr{F}')$ on a ground set of doubled size $|E'|=2\cdot|E|$ and a surjective map $E'\to E$, which induces an isomorphism of posets $\mathscr{F}'\to\mathscr{F}$. Thus rank and meet are preserved by this map, however intersection is not. As basis graph, basic word graph, connectivity and diameters are poset properties, $\mathscr{F}$ and $\mathscr{F}'$ can be considered equivalent for our purposes, i.e. we regard undirected branchings as a special case of directed branchings. Compare Figure 1.

Now let V be the set of vertices of D different from the root, and $V(\mathscr{F})$ the set of vertex sets in V of trees in $\mathscr{F}$. Then $V(\mathscr{F})$ is again a greedoid, the *vertex search greedoid* on D [2]. In fact it is an upper interval greedoid, because the union of two feasible vertex sets is obviously again feasible.

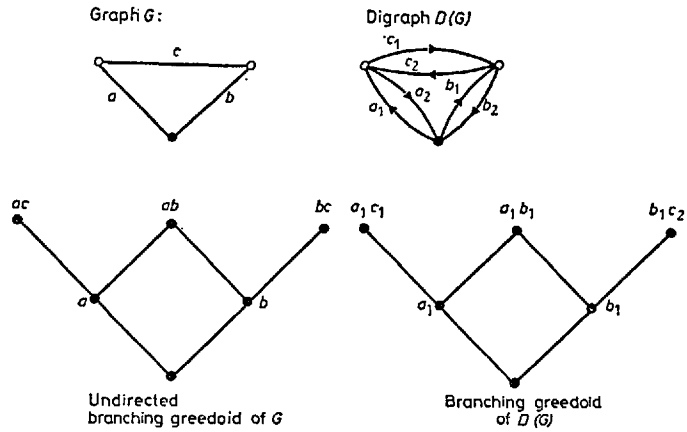

Fig. 1

In Section 4 we will analyze the structure of branching and vertex search greedoids as well as their relationship in detail. Compare [12] for characterization results.

A special case of upper interval greedoids are the *poset greedoids* $(E, \mathcal{F})$ for which $\mathcal{F}$ is also closed under intersection. They all arise as the set of order ideals in a partial order on E. In this case $\mathcal{F}$ is a distributive lattice, which again characterizes poset greedoids.

Finally *local poset greedoids* are greedoids such that every interval in $\mathcal{F}$ is a distributive lattice. Note that every greedoid $\mathcal{F}$ contains the minimal element $\emptyset$, but need not be a meet semi-lattice in general. For interval greedoids, however, $\mathcal{F}$ is a meet semilattice, and all local poset greedoids are interval greedoids. Examples of local poset greedoids are poset greedoids, branching greedoids, and matroids.

3. Two-connected greedoids

Korte and Lovász proved that for a 2-connected greedoid $\mathcal{F}$ both G and $\hat{G}$ are connected. In [4], Björner, Korte and Lovász raise the diameter question for these graphs. In particular they show that the diameter of G can exceed the rank of $\mathcal{F}$. In fact L. Lovász described a class of 2-connected undirected branching greedoids whose diameter grows quadratically with the rank. We give a refined version of his construction in Section 4. However, for general 2-connected greedoids of rank r we now show that the maximal diameters of basis graphs and basic word graphs are exponential:

Theorem 3.1. Let $\mathcal{F}$ be a 2-connected greedoid of rank r. Then $\operatorname{diam} G(\mathcal{F}) \leq 2^r - 1$, $\operatorname{diam} \hat{G}(\mathcal{F}) \leq 3 \cdot 2^r - 2r - 3$, and both bounds are best possible.

Corollary 3.2. (Korte and Lovász [10].) Let $\mathcal{F}$ be a 2-connected greedoid of rank r. Then the basis graph $G(\mathcal{F})$ and the basic word graph $\hat{G}(\mathcal{F})$ are connected.

For the proof of Theorem 3.1 we use two lemmas which concern the case $r=2$.

Lemma 3.3. Let $(E, \mathcal{F})$ be 2-connected, $X \in \mathcal{F}$ of corank at least 2. Let $X \cup \{x\}$ be feasible, then there is a $y \in E \setminus X$, $x \neq y$, such that $\{x, y\}$ is free over X.

Proof. Let $y, z \in E \setminus X$ such that $\{y, z\}$ is free over X. Now $X \cup \{x\}$ can be augmented from $X \cup \{y, z\}$. Without loss of generality $X \cup \{x, y\} \in \mathcal{F}$, i.e. $\{x, y\}$ is free over X. ∎

Lemma 3.4. Let $\{x, y\}$, $\{z, w\}$ be bases of a 2-connected greedoid of rank 2. Then $d(\{x, y\}, \{z, w\}) \leqq 3$.

Proof. We assume that $\{x\}, \{z\} \in \mathcal{F}$ and $\{x, z\} \notin \mathcal{F}$ (so $x \neq z$). By Lemma 3.3, there is a $u \in E$, $u \neq x$, such that $\{u\}, \{x, u\} \in \mathcal{F}$ (thus $u \neq z$). Now $\{z\}$ can be augmented from $\{x, u\}$. Thus $\{z, u\}$ is a basis of $\mathcal{F}$ and we are done (compare to Figure 2*a*). ∎

Proof of Theorem 3.1. For the proof we need some notation. Let $d(r)$ (respectively $\hat{d}(r)$) be the maximal diameter of the basis graph (respectively basic word graph) of a 2-connected greedoid of rank r. For $\{x\}$ feasible let $\bar{x}$ be the set of bases that contain $x \in E$ as an element and let $\delta(r)$ be the maximal distance between a basis X and a set $\bar{x}$ in any 2-connected greedoid of rank r. Similarly let $\hat{x}$ be the set of basic words that contain $x \in E$ as their first letter and let $\hat{d}(r)$ be the maximal distance between a basic word ξ and a set $\hat{x}$ in any 2-connected greedoid of rank r. The existence (finiteness) of of these diameters and distances will follow from the recursions below which at the same time establish our upper bounds.

Now we have to prove that $d(r) = 2^r - 1$ and $\hat{d}(r) = 3 \cdot 2^r - 2r - 3$. Note that Lemma 3.4 shows that $d(2) \leqq 3$, $\hat{d}(2) \leqq 5$, where a greedoid exhibiting this "worst case" is shown in Figure 2a. This proves the case $r=2$.

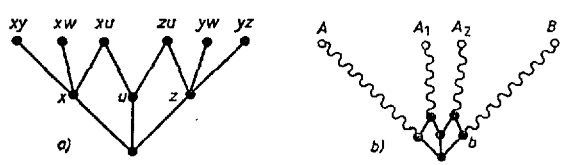

Fig. 2

Now the situation for $r=2$ allows to derive the following recursions by the argument indicated in Figure 2b:

$$d(r+1) \leqq \delta(r+1) + d(r); \quad \delta(r+1) \leqq 2 \cdot \delta(r);$$

$$\hat{d}(r+1) \leqq \hat{\delta}(r+1) + \hat{d}(r); \quad \hat{\delta}(r+1) \leqq 2 \cdot \hat{\delta}(r) + 2;$$

which together with the trivial initial conditions proves

$$\delta(r) \leqq 2^{r-1}; \quad d(r) \leqq 2^r - 1;$$

$$\hat{\delta}(r) \leqq 3 \cdot 2^{r-1} - 2; \quad \hat{d}(r) \leqq 3 \cdot 2^r - 2r - 3.$$

To prove that these bounds are actually sharp, we now construct inductively an interval greedoid G_r for every r, which attains them. We shall have $G_r=([2r];\mathcal{F})$ with $|\mathcal{F}|=3\cdot 2^r-r-2$. Put $G_0=(\emptyset, \{\emptyset\})$, $G_1=([2], \{\emptyset, \{1\}, \{2\}\})$. Define a k-set $\{a_1, a_2, ..., a_k\}$ to be regular if $a_i\in\{2i-1, 2i\}$ for all i. Now define the greedoid G_{r+1} to consist of the following feasible sets:
(1) the feasible sets of G_r
(2) the $(r-1)$-sets of G_r augmented by $2r+1$ if they are regular, by $2r+2$ otherwise
(3) as bases ($(r+1)$-sets) the bases of G_r, augmented by $2r+1$ or $2r+2$, i.e. all regular r-sets.

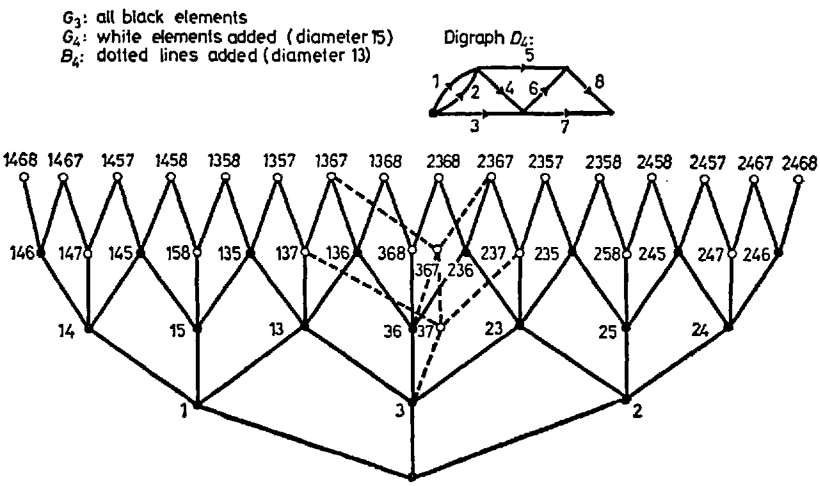

Fig. 3

Now we check that each G_r actually is a 2-connected interval greedoid (compare Figure 3). G_r is an accessible set system by construction. To verify the exchange axiom, it is sufficient to consider $X, Y\in G_r$, $|X|=|Y|+1$. By symmetry between 1 and 2 in G_r it is sufficient to consider the following cases: if 1 is contained in both X and Y, then we can reduce to the contracted greedoid $G_r/\{1\}$, which is isomorphic to G_{r-1}; there the exchange property holds by induction. If $1\in X$, $2\in Y$, then we can augment $Y\setminus\{2\}$ from $X\setminus\{1\}$ in $G_r/\{1\}=G_{r-1}$ by induction. If $1\in X$ and $\{1, 2\}\cap Y=\emptyset$, then $Y\cup\{1\}\in\mathcal{F}$. And for $1\in Y$, $\{1, 2\}\cap X=\emptyset$ it is sufficient to augment Y from $X\cup\{1\}\in\mathcal{F}$, which again can be done by induction. Finally, if 1 and 2 are contained in neither X nor Y, then by construction $Y\subseteq X$ and we are done.

Two-connectivity follows easily by induction on r, because at each step it has to be checked only for sets of corank 2.

To verify the interval property, one can observe (again by induction on r) that the intervals of G_r are distributive lattices of width at most two. Thus each G_r is even a local poset greedoid, however not a branching greedoid for $r>3$ (see Section 4). G_r has 2^r bases (the regular r-sets), and the basis graph $G(G_r)$ is a chain; thus $d(r)=2^r-1$. Actually:

$$d(\{1, 4, 6, ..., 2r\}, \{2, 4, ..., 2r\}) = 2^r-1,$$

and

$$\delta(r) = d(\{1, 4, 6, \dots, 2r\}, \bar{2}) = 2^{r-1} \quad \text{for} \quad r \geqq 1.$$

Also by construction we find that the inequalities for $\hat{d}$ and $\hat{\delta}$ are actually sharp here, so

$$\hat{d}(146 \cdot \dots \cdot 2r, 246 \cdot \dots \cdot 2r) = 3 \cdot 2^{r-1} - 2r - 3$$

$$\hat{\delta}(r) = \hat{d}(146 \cdot \dots \cdot 2r, \hat{2}) = 3 \cdot 2^{r-1} - 2 \quad \text{for} \quad r \geqq 1.$$

where "$\hat{d}$" denotes distance in the basic word graph. ∎

We remark that the greedoid G_r constructed in the proof of 3.1 is a "slimming" (in the sense of Korte and Lovász [9]) of the matroid of the r-dimensional hyperoctahedron. However none of the slimming methods described in [9] yields a construction of our G_r. The refined "trimming" methods described in [11] do not apply either; they do not even produce local poset greedoids.

Note that there is an alternative way for deriving the upper bound for $d(r)$ in Theorem 3.1 by showing part (1) of the following theorem. However this point of view is less helpful in the study of branching greedoids (Section 4).

Theorem 3.5. Let $\mathscr{F}$ be a 2-connected greedoid of rank $r (r \geqq 1)$. Then

(1) $$\operatorname{diam} G(\mathscr{F}) \leqq 2 \cdot \operatorname{diam} G(\mathscr{F}|_{r-1}) + 1.$$

If $\mathscr{F}$ is an interval greedoid then

(2) $$\operatorname{diam} G(\mathscr{F}|_{r-1}) \leqq (r-1) \cdot (\operatorname{diam} G(\mathscr{F}) + 1).$$

If $\mathscr{F}$ is a local poset greedoid, then the following stronger bound holds:

(3) $$\operatorname{diam} G(\mathscr{F}|_{r-1}) \leqq \operatorname{diam} G(\mathscr{F}) + 1.$$

Proof. The first inequality is a straightforward consequence of Lemma 3.4. For (2) we observe that given any basis B in $\mathscr{F}$, the restriction $[\emptyset, B]$ of $\mathscr{F}$ is an upper interval greedoid, hence the basis graph of its $(r-1)$-truncation is connected by Lemma 2.8. This graph has at most r vertices, thus diameter at most $r-1$. This implies (2). For (3) we observe that for local poset greedoids, the intervals are distributive and hence modular lattices. Thus for two subbases $\bar{X}$ and $\bar{Y}$ of $\mathscr{F}$, their union $\bar{X} \cup \bar{Y}$ is a basis of $\mathscr{F}$ if and only if $\bar{X} \cap \bar{Y}$ is a subbasis of $\mathscr{F}|_{r-1}$. Thus $\bar{G}(\mathscr{F})$ and $G(\mathscr{F}|_{r-1})$ coincide. With Lemma 2.6 we are done. ∎

Theorem 3.1 shows that (1) is actually sharp. This is not the case for (2) and (3).

Observe that the proof of Theorem 3.1 actually yields an algorithm of complexity roughly $\hat{d}(r)$ to construct a path in the basis graph between any given pair of bases. To understand the design and complexity of such algorithms in general, it seems natural to ask how diam $\hat{G}(\mathscr{F})$ is bounded (from above and below) in terms of the diameters diam $G(\mathscr{F}|_i)$ for $1 < i \leqq r$. This appears to be equivalent to understanding the structures of shortest paths in $\hat{G}$. It turns out that the inequalities suggested by the extremal examples that we construct in this paper do not hold in general.

4. Two-connected branching greedoids

For proving bounds on the diameter of branching greedoids, it will be convenient to work with the poset of flats, which turns out to have more global structural properties than the poset of feasible sets. The poset of flats associated with any greedoid was introduced and studied by Björner [2]. We refer to his analysis for details. The basic construction is as follows. Let $(E, \mathcal{F})$ be a greedoid. Define an equivalence relation on $\mathcal{F}$ by $X \approx Y$ iff X and Y have the same continuations, that is iff $\mathcal{F}/X = \mathcal{F}/Y$. The set $L(\mathcal{F})$ of equivalence classes has a partial order induced by the inclusion order on $\mathcal{F}$. This poset is called the *poset of flats* of $\mathcal{F}$.

Note that this construction is compatible with contraction, that is the poset of flats of a contraction is identical to the corresponding upper interval in the poset of flats. This fact together with the observation that any contraction of a branching greedoid is again a branching greedoid makes the proof by induction on r for Theorem 4.6 possible.

We will need several lemmas:

Lemma 4.1. $L(\mathcal{F})$ is graded with $\hat{0}$ and $\hat{1}$. The quotient map $\Phi\colon \mathcal{F} \to L(\mathcal{F})$ is order preserving, rank preserving and surjective. If X_0 is a chain in $\mathcal{F}$ and $\mathbf{c}$ is a maximal chain in $L(\mathcal{F})$ such that $\Phi(X_0) \subseteq \mathbf{c}$, then there is a maximal chain X in $\mathcal{F}$ such that $\Phi(X) = \mathbf{c}$. This especially implies that Φ is also surjective as a map of chains.

Proof. The first two assertions are immediate from the definitions. The third is clear from the description of $L(\mathcal{F})$ as a poset representation of $\mathcal{F}$ in the sense of Björner [2]. ∎

Lemma 4.2. If $\mathcal{F}$ is k-connected, then so is $L(\mathcal{F})$. If $\mathcal{F}$ is an interval greedoid, then the converse is also true.

Proof. The first statement is obvious. For the second we consider the following edge labeling of the Hasse diagram of $L(\mathcal{F})$, which makes $L(\mathcal{F})$ into a poset representation of $\mathcal{F}$ (see [2]). For any cover relation $\bar{X} \lessdot \bar{Y}$ in $L(\mathcal{F})$ put $\lambda(\bar{X} \lessdot \bar{Y}) = \{x \in E\colon X \cup \{x\} \in \bar{Y} \text{ for } X \in \bar{X}\}$. The reconstruction of free k-sets in $\mathcal{F}$ is easily done using that if $\bar{X}$ is covered by $\bar{Y}$ and $\bar{Z}$ in $L(\mathcal{F})$ and they are both covered by $\bar{W} (\bar{Y} \neq \bar{Z})$, then $\lambda(\bar{X} \lessdot \bar{Y}) \subseteq \lambda(\bar{Z} \lessdot \bar{W})$. This is clear from the strong exchange property for interval greedoids [2]: we can use an argument similar to that in [4, proof of Thm. 3.6.]. Note that $\lambda(\bar{X} \lessdot \bar{Y}) \cap \lambda(\bar{X} \lessdot \bar{Z}) = \emptyset$. ∎

Lemma 4.3. If $\mathcal{F}$ is an interval greedoid, then $L(\mathcal{F})$ is a semimodular lattice. In this case $L(\mathcal{F})$ is coatomic if and only if $\mathcal{F}$ is 2-connected.

Proof. The first part is a result by Crapo [5], which Björner [2] translates into the language of greedoids. For another proof see [4]. We remark that the converse is wrong. For the second part, $L(\mathcal{F})$ is coatomic if and only if every element of corank greater than one has at least two covers. In a semimodular lattice this is equivalent to 2-connectedness. By Lemma 4.2, we are done. ∎

Now let $(E, \mathcal{F})$ be a branching greedoid of rank r in a digraph D, as defined in Section 2, then $(V, L(\mathcal{F}))$ is the vertex search greedoid on D, as Björner observed in [2], that is the lattice of flats of the branching greedoid $L(\mathcal{F})$ is naturally isomorphic to the poset of feasible sets $V(\mathcal{F})$ of the corresponding vertex search greedoid. Together with Lemma 4.2, this implies that the two greedoids have the same connectivity.

In the case of directed branching greedoids the projection map Φ is actually induced by the map Φ_0: $E \to V$ that sends every arc x to its endpoint (head) $\Phi_0(x)$ such that $\Phi(X)=\{\Phi_0(x)|x\in X\}$: the image of a tree is its set of vertices. For this we can assume that the root is not the endpoint of any arc in E.

The following lemma plays a key role in our proof for Theorem 4.6. Note that it fails if we weaken the hypotheses to local poset greedoids, e.g. for the greedoid G_4 constructed in Section 3.

Lemma 4.4. Let $\mathcal{F}$ be a 2-connected branching greedoid of rank $r\geqq 2$, $L(\mathcal{F})$ its lattice of flats. Let a_1, b_1 be two distinct atoms in $L(\mathcal{F})$. Then $L(\mathcal{F})$ contains two maximal chains $\mathbf{a}$: $\hat{0}<a_1<a_2<\dots<a_{r-1}<\hat{1}$ and $\mathbf{b}$: $\hat{0}<b_1<b_2<\dots<b_{r-1}<\hat{1}$ such that a_i and b_{r-i} are complements for $1\leqq i\leqq r-1$.

Proof. We construct the a_i inductively as feasible subsets of V, such that their complements in V are feasible sets of size $r-i$. For a_1 we can choose a complementing coatom b_{r-1} because $L(\mathcal{F})$ is coatomic. Note that complements of complementary rank in $L(\mathcal{F})$ are unique if they exist.

Now assume that a_i is constructed, and let $A_i\in\mathcal{F}$ be a spanning tree for a_i (i.e. $A_i\in\mathcal{F}$, $\Phi(A_i)=a_i$). Then $\mathcal{F}/A_i$ is a 2-connected greedoid of rank $r-i$. Let $\beta_1\in E$ be an arc from the root to the vertex b_1, and let B_{r-i} be a spanning tree for b_{r-i} that contains β_1. Then B_{r-i} is a basis of $\mathcal{F}/A_i$.

The greedoid $\mathcal{F}/A_i$ contains atoms like $\{\beta_1\}$ which are also feasible in $\mathcal{F}$, and in the non-trivial case (otherwise we are done by induction on r) it also contains atoms — call them "red" — which are not feasible in $\mathcal{F}$. These correspond to "red" arcs in D from vertices in a_i (not the root) to vertices in $V\setminus a_i=b_{r-i}$. We choose a basis A^0 of $\mathcal{F}/A_i$ that contains a red arc. Now the basis graph of $\mathcal{F}/A_i$ is connected by Corollary 3.2. On a path from B_{r-i} to A^0 in this basis graph let B^0 be the first basis that contains a red arc x. Then $B_{r-i-1}=B^0\setminus\{x\}$ is a feasible set of rank $r-i-1$ in $\mathcal{F}/A_i$ by construction. B_{r-i-1} is actually in $\mathcal{F}$ because it does not contain any red arc. Thus we can define a_{i+1} as $a_i\cup\Phi_0(x)$ and b_{r-i-1} as $\Phi(B_{r-i-1})$ to complete the induction step. ∎

Note that the complementary chains constructed in this lemma are in general not unique if $r\geqq 4$.

It would be nice to give a lattice theory version of Lemma 4.4, stating that a collection of lattice properties of $L(\mathcal{F})$ implies the existence of complementary chains through given atoms. Several such properties are obvious or already verified: $L(\mathcal{F})$ is coatomic (2-connected), primary (the join irreducibles form an order ideal), and semimodular. However these properties together do not force the existence of complementary chains, as pointed out to me by M. Haiman. Also $L(\mathcal{F})$ is a locally free lattice, which allows to consider it as an upper interval greedoid. In this situation a suitable strengthening of the condition of being primary might allow to give a proof. However, in view of the characterization results on branching greedoids by W. Schmidt [12] it is not clear whether such a proof could yield a result that is in fact more general than Theorem 4.6.

Lemma 4.5. Let $\mathscr{F}$ be a 2-connected branching greedoid, $\mathbf{a}$ and $\mathbf{b}$ complementary chains in $L(\mathscr{F})$ as constructed in 4.4 and X_0, Y_0 chains in $\mathscr{F}$ such that $\Phi(X_0)\subseteq\mathbf{a}$ and $\Phi(Y_0)\subseteq\mathbf{b}$. Then there are maximal chains X, Y in $\mathscr{F}$ such that $\Phi(X)=\mathbf{a}$, $\Phi(Y)=\mathbf{b}$ and $X_i\cup Y_j$ is feasible of rank $i+j$ for $i+j\leqq r$.

Proof. Clear. ∎

For 2-connected branching greedoids of rank r, let $b(r)$ be the maximal diameter of the basis graph, and similarly $\bar{b}(r)$ and $\hat{b}(r)$ for the subbasis graph and the basic word graph. Now from Theorem 3.1 we know that the diameter of 2-connected branching greedoids of rank r is bounded by $b(r)\leqq 2^r-1$. The greedoids constructed there as extremal examples are, however, not branching greedoids for $r>3$. It is somewhat surprising to see that $b(r)$, the diameter of branching greedoids, is quadratically bounded in the rank, and the exact bound can again be determined:

Theorem 4.6. Let $\mathscr{F}$ be a 2-connected branching greedoid of rank r. Then $\operatorname{diam} G(\mathscr{F})\leqq r^2-r+1$ and

$$\operatorname{diam}\hat{G}(\mathscr{F})\leqq\binom{r+2}{3}-2r+1.$$

Both bounds are sharp for $r>0$.

Proof. We first prove $\bar{b}(r)\leqq r^2-r$, which implies the bound on $\operatorname{diam} G(\mathscr{F})$ by Lemma 2.6. The proof for the basic word graphs is similar to that for the basis graphs and will be omitted (compare the proof for Theorem 3.1).

Let $(E,\mathscr{F})$ be a 2-connected branching greedoid of rank r, $\mathscr{L}$ the corresponding language. Arcs emanating from the root (corresponding to feasible elements in E or 1-sets in $\mathscr{F}$) will be called *stems*.

Let $X=x_1x_2...x_{r-1}$ and $Y=y_1y_2...y_{r-1}$ be subbasic words of $\mathscr{L}$ (i.e. X and Y are words of length $r-1$ in the greedoid language $\mathscr{L}$ corresponding to $\mathscr{F}$ such that $X_i=\{x_1, x_2, ..., x_i\}$ and $Y_i=\{y_1, y_2, ..., y_i\}$ are feasible sets in $\mathscr{F}$ for $1\leqq i\leqq r-1$. Especially $\tilde{X}=X_{r-1}$ and $\tilde{Y}=Y_{r-1}$ are subbases of $\mathscr{F}$). We can assume $x_1\neq y_1$ by induction on r. To get a recursive bound on the possible distance between $\tilde{X}$ and $\tilde{Y}$ in $\bar{G}$ we distinguish several cases which depend on the particular choice of X and Y. Let $\bar{b}_1(r)$ be the maximal distance $\bar{d}(\tilde{X},\tilde{Y})$ in $\bar{G}$ in the case that x_1 and y_1 are parallel stems, that is if $\Phi(X_1)=\Phi(Y_1)$. Let $\bar{b}_2(r)$ denote the maximal distance of $\tilde{X}$ and $\tilde{Y}$ in the case that x_1 and y_1 are independent, i.e. if $\Phi(X_1)\neq\Phi(Y_1)$. By $\bar{b}_3(r)$ we denote the maximal distance of $\tilde{X}$ and $\tilde{Y}$ in the case that $\Phi(X_1)$ and $\Phi(\tilde{Y})$ are complements in $L(\mathscr{F})$, which especially implies that $\Phi(X_1)\neq\Phi(Y_1)$. Finally $\bar{b}_4(r)$ will be the distance between $\tilde{X}$ and $\tilde{Y}$ in the case that the $a_i=\Phi(X_i)$ and the $b_i=\Phi(Y_i)$ form complementary chains in $L(\mathscr{F})$ in the sense of Lemma 4.4, which in turn implies that $\Phi(X_1)$ and $\Phi(\tilde{Y})$ are complements. We observe that $\bar{b}_4(r)=r-1$ by construction, and $\bar{b}_2(r)\geqq\bar{b}_3(r)$ is obvious.

We want to show that $\bar{b}_3(r)\leqq\bar{b}_3(r-1)+\bar{b}_4(r)$. For this, take X and Y as above and let $a_1=\Phi(X_1)$. $b_1=\Phi(Y_1)$. Now, by Lemma 4.4, we can construct complementary chains $\mathbf{a}$ and $\mathbf{b}$ in $L(\mathscr{F})$ containing a_1 and b_1, such that $\Phi(\tilde{Y})=b_{r-1}$. The coatoms of the join-semilattice generated in $L(\mathscr{F})$ by $\mathbf{a}$ and $\mathbf{b}$ are exactly the $s_i=a_i\cup b_{r-1-i}$ for $0\leqq i\leqq r-1$, and these are distinct. It is easy to see that the meet of the set of coatoms $\{s_i\}$ in the lattice $L(\mathscr{F})$ is $\hat{0}$. Now $\Phi(X_2)\leqq b_{r-1}=s_0$ is impossible

Thus there is an s_i with $0<i\leqq r-1$ such that $\Phi(X_2)\cup s_i=\hat{1}$, hence by semimodularity $\Phi(X_2)$ and s_i are complements in the upper interval $[a_1, \hat{1}]$ of $L(\mathscr{F})$. Now by choosing $\tilde{Z}\in\Phi^{-1}(s_i)$ consistently with X_1 and $\tilde{Y}$ (as described by Lemma 4.5), we find that $\bar{d}(\tilde{X}, \tilde{Z})\leqq\bar{b}_3(r-1)$ and $\bar{d}(\tilde{Z}, \tilde{Y})\leqq\bar{b}_4(r)$. This shows $\bar{b}_3(r)\leqq\bar{b}_3(r-1)+$ $+\bar{b}_4(r)$, and thus

$$\bar{b}_3(r) \leqq \binom{r}{2}.$$

Now for getting a bound on $\bar{b}_1(r)$, let $\tilde{X}$ and $\tilde{X}'$ be given with feasible permutations X and X' such that $\Phi(X_1)=\Phi(X_1')$. We can choose a subbasis $\tilde{Y}$ such that $\Phi(X_1)$ and $\Phi(\tilde{Y})$ are complements in $L(\mathscr{F})$. Then we have $\bar{d}(\tilde{X}, \tilde{X}')\leqq\bar{d}(\tilde{X}, \tilde{Y})+\bar{d}(\tilde{Y}, \tilde{Y}')$ and thus $\bar{b}_1(r)\leqq 2\cdot\bar{b}_3(r)=r^2-r$.

To estimate $\bar{b}_2(r)$, let $\tilde{X}$, $\tilde{Y}$, X and Y be given as above. Then we construct complementary chains through $\Phi(X_1)$ and $\Phi(Y_1)$ by Lemma 4.4. Now the same argument we used for bounding $\bar{b}_3(r)$ shows that

$$\bar{b}_2(r) \leqq \bar{b}_3(r-1)+\bar{b}_4(r)+\bar{b}_3(r-1) \leqq r^2-2r+1.$$

But this finally implies

$$\bar{b}(r) \leqq \max\left(\bar{b}_1(r);\ \bar{b}_2(r)\right) \leqq r^2-r \quad (r \geqq 1).$$

To prove optimality of the bound, we construct extremal branching greedoids: Let B_r for $r\geqq 0$ be the branching greedoid of rank r on the digraph D_r, which has vertex set $\{v_0, v_1, \ldots, v_r\}$, root $v_{-1}=v_0$, arcs labeled "$2i-1$" ($1\leqq i\leqq r$) from v_{i-2} to v_i and arcs labeled "$2i$" ($1\leqq i\leqq r$) from v_{i-1} to v_i (compare Figure 4).

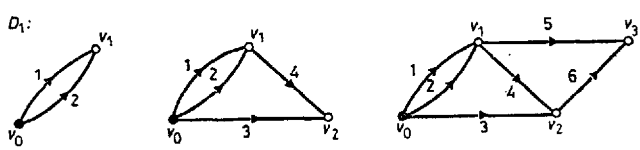

Fig. 4

We observe that the bases of B_r are exactly the regular r-sets, i.e. the bases of G_r as defined in Section 3. In fact the greedoids satisfy $B_r=G_r$ for $r\leqq 3$, and G_r is a proper subgreedoid of B_r for $r\geqq 4$ ($\{3, 7\}$ and $\{3, 6, 7\}$ are feasible in B_r, but not in G_r for $r\geqq 4$). The distance $b_r=d(\{1, 4, 6, \ldots, 2r\}, \{2, 4, 6, \ldots, 2r\})$ can be computed inductively by noting that for $r>1$, one has to pass through a basis containing 3, but not 5. Let β_r be the minimal distance from $\{1, 4, 6, \ldots, 2r\}$ to such a basis. Then we have $b_r=2\cdot\beta_r+1$ (equality because of the symmetry between 1 and 2), and $\beta_{r+1}=\beta_r+r$ by the obvious induction step. Thus we get

$$\beta_r \geqq \binom{r}{2} \quad \text{and} \quad b(r) \geqq 2\cdot\binom{r}{2}+1 = r^2-r+1. \quad \blacksquare$$

Theorem 4.6 answers the question by L. Lovász mentioned in the introduction (Section 1).

We have a few obvious corollaries. First, if we exclude parallel arcs in the digraph, we get a class of greedoids, on which the diameter never exceeds $\bar{b}_2(r)+1$, for which we determined r^2-2r+2 as an upper bound. This bound is not sharp for $r\geqq 4$, but it differs from the minimal bound only by a term of linear order of magnitude. However we remark that the condition "without parallel arcs" does not seem natural, as it is not preserved under contractions. If we exclude antiparallel arcs the bound decreases further.

The upper bound for the diameter is the same for directed and undirected branching greedoids. In fact the undirected branching greedoids on the graphs D_r have the same diameters as the corresponding directed branching greedoids; on the other hand our discussion in Section 2 shows that every undirected branching greedoid corresponds to a directed branching greedoid of the same rank and diameter.

For both Theorem 3.1 and Theorem 4.6 the extremal examples are not unique. Different greedoids of the same diameter can be constructed by duplicating edges in D_r respectively by adding the r-sets of G_{r+1} to G_r. However, the following holds:

Proposition 4.7. Let $\mathcal{F}$ be a 2-connected greedoid (respectively 2-connected branching greedoid) of rank r and diameter $d(r)$ (respectively $b(r)$). Then $\mathcal{F}$ contains G_r (respectively B_r) as a subgreedoid.

Proof. The statement for the case of general greedoids follows from a construction by induction on the rank using Lemma 3.3 in the same way as for Theorem 3.4 (1). For the case of branching greedoids it is possible to reconstruct a subgraph D_r in the graph of a given greedoid using induction and the fact that in the proof of Theorem 4.6 the recursive inequalities for $\bar{b}_1$ and $\bar{b}_3$ are sharp. ∎

In the context of branchings in rooted graphs and the diameter problem for them it seems natural to ask the following question, which gives an unrooted analogue in the case of undirected graphs:

Let $G=(V, E)$ be a 2-connected, undirected graph on $r+1$ vertices (rank r). How many exchanges of leaves are at most necessary to transform a spanning tree into a given second one?

As in the rooted case this problem translates into the diameter question for the graph whose vertices are the spanning trees of G (i.e. the bases of the graphic matroid), and in which two spanning trees are joined by an edge if they differ in a single leaf.

Now we can choose an arbitrary vertex as a root to see from Theorem 3.1 that the diameter of this graph (the maximal number of steps necessary) is not larger than r^2-r+1.

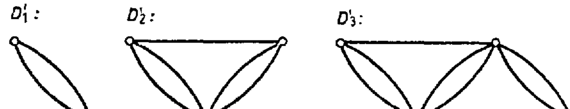

Fig. 5

On the other hand Figure 5 shows a sequence of graphs which yield lower bounds on the diameters that are also quadratic (roughly half as large): $2\cdot b(r)=$ $=2r^2-2r+2$ for the graphs of rank $2r$, and $b(r+1)+b(r)=2r^2+r$ in the case of rank $2r+1$. We conjecture that these graphs are extremal examples.

5. Higher connectivity

For the case of greedoids with higher connectivity, it is necessary to assume the interval property. This is so because the following key lemma (which generalizes Lemma 3.3) is no longer true for $k\geqq 3$ without assuming the interval property:

Lemma 5.1. Let $(E, \mathcal{F})$ be a k-connected interval greedoid. If $X\in\mathcal{F}$ has corank at least k, and $X\cup\{x\}$ is feasible $(x\in E\setminus X)$, then there is a free k-set A over X which contains x. Moreover, if any free k-set B over X is given, then A can be chosen such that $A\setminus B\subseteq\{x\}$.

Proof. We can assume that $X=\emptyset$ and that $\mathcal{F}$ has rank k. Thus $\{x\}$ is feasible, and there is a free k-set (basis) B. We assume $x\notin B$. Now by successive augmentations of $\{x\}$ from B we find that there is a basis $A'\cup\{x\}$ of $\mathcal{F}$ such that $A'\subseteq B$. But every subset of A' is feasible, thus by interval property find that $A'\cup\{x\}$ is free. ∎

We use a second lemma:

Lemma 5.2. In an interval greedoid $\mathcal{F}$ of rank r let X be a basis and Y a free basis. Then the distance between X and Y (in the basis graph) is at most r.

Proof. By induction on r, using Lemma 5.1. ∎

These two lemmas allow to solve the maximally connected case completely, which in turn can be used to prove bounds for "intermediate cases". In particular Lemma 5.2 implies that for r-connected (i.e. maximally connected) greedoids the basis graph has at most radius r.

Theorem 5.3. Let $(E, \mathcal{F})$ be an r-connected interval greedoid of rank r $(r\geqq 1)$. Then $\operatorname{diam} G(\mathcal{F})\leqq 2r-1$, and this bound is the best possible.

Proof. Let X and Y be bases. If Z is a free basis containing a feasible element x_1 of X (such a Z exist by Lemma 5.1), then $d(X, Z)\leqq r-1$ and $d(Z, Y)\leqq r$ by Lemma 5.2. This proves the bound. Its optimality is easily proved by construction of a sequence of suitable branching greedoids (Figure 6). ∎

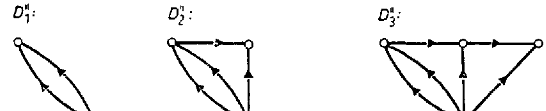

Fig. 6

It is interesting to compare this result to the entirely trivial cases. The diameter of a matroid is obviously bounded by is rank, for an antimatroid the diameter is 0 (Lemma 2.8.).

Also it is easy to determine the maximal diameter for the corresponding basic word graphs. The values are r^2+r-1 for r-connected interval greedoids, $\binom{r+1}{2}$ for matroids, and $\binom{r}{2}$ for antimatroids. This last result also appears in [1] in connection with the discussion of an analogue of the Hirsch conjecture for semimodular lattices.

Theorem 5.3 especially shows that the diameter of 3-connected interval greedoids of rank 3 is at most 5, thus lower than the maximal value for 2-connected greedoids of the same rank. In fact the same is true for greedoids without the interval property, but the proof we know for this is a pretty tedious case by case analysis. Thus also for 3-connected interval greedoids of higher rank the possible diameter is smaller than $d(r)$. The key question seems to be whether for 3-connected interval greedoids the diameter is polynomially bounded in the rank. This would for example follows if the conclusion of Lemma 4.4 were true for any 3-connected lattice.

We give one relatively straightforward general result:

Theorem 5.4. Let $(E, \mathscr{F})$ be a k-connected interval greedoid of rank r $(r \geqq k \geqq 2)$. Then diam $G(\mathscr{F}) \leqq k \cdot 2^{r-k+1}-1$.

Proof. We use induction on $r-k$, the case $r=k$ being clear.

Let X and Y be bases, x_1 and y_1 feasible elements in them. Now if $\{X \in \mathscr{F}: x_1 \in X\}$ and $\{Y \in \mathscr{F}: y_1 \in Y\}$ intersect (i.e. $\{x_1, y_1\} \in \mathscr{F}$ by interval property), in which case they have a basis Z in common, then we get

$$d(X, Y) \leqq d(X, Z)+d(Z, Y) \leqq 2 \cdot (k \cdot 2^{(r-1)-k+1}-1) < k \cdot 2^{r-k+1}-1$$

by induction. Now there is a feasible element z_1 such that $\{x_1, z_1\}$ and $\{z_1, y_1\}$ are both feasible. In fact, by repeated application of the argument that proves Lemma 3.4 we may conclude that there are feasible sets Z_i for $1 \leqq i < r$ such that $\{x_1\} \cup Z_i = X'$ and $Z_i \cup \{y_1\} = Y'$ are both feasible. Especially this means that there is a subbasis Z_{r-1} such that $\{x_1\} \cup Z_{r-1} = X'$ and $Z_{r-1} \cup \{y_1\} = Y'$ are both bases. Thus we get from the induction hypothesis:

$$d(X, Y) \leqq d(X, X')+d(X', Y')+d(Y', Y) \leqq$$
$$\leqq (k \cdot 2^{(r-1)-k+1}-1)+1+(k \cdot 2^{(r-1)-k+1}-1) = k \cdot 2^{r-k+1}-1. \quad \blacksquare$$

Theorem 5.4 reduces to the upper bounds of 3.1 for $k=2$ and of 5.3 for $k=r$. In these two cases Theorem 5.4 is best possible. However, this is not true in general for $r > k > 2$. In fact using Theorem 5.3 and Lemma 5.2 for $k=3$, it is easy to see that the diameter of 3-connected interval greedoids of rank 5 is never larger than 17. The result generalizes without hinting to a uniform better bound.

In the case of branching greedoids, Theorem 5.3 still applies. It is not hard to see that for fixed connectivity the diameter of branching greedoids is bounded by a quadratic polynomial in the rank, with the leading coefficient decreasing if k is increased. For example, in the base of 3-connected branching greedoids the diameter seems to be bounded by $r^2/2+1$.

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